Exercise 30

Let

$$g(x) = \begin{cases} 2x - x^2 & \text{if } 0 \le x \le 2 \\ 2 - x & \text{if } 2 < x \le 3 \\ x - 4 & \text{if } 3 < x < 4 \\ \pi & \text{if } x \ge 4 \end{cases}$$

(a) For each of the numbers 2, 3, and 4, discover whether g is continuous from the left, continuous from the right, or continuous at the number.

(c) Sketch the graph of g.

Solution

Evaluate all of the limits as $x \to 2$, $x \to 3$, and $x \to 4$.

$$\lim_{x \to 2^-} g(x) = \lim_{x \to 2}(2x - x^2) = 2(2) - (2)^2 = 0$$

$$\lim_{x \to 2^+} g(x) = \lim_{x \to 2}(2 - x) = 2 - 2 = 0$$

$$\lim_{x \to 2} g(x) = 0$$

$$\lim_{x \to 3^-} g(x) = \lim_{x \to 3}(2 - x) = 2 - 3 = -1$$

$$\lim_{x \to 3^+} g(x) = \lim_{x \to 3}(x - 4) = 3 - 4 = -1$$

$$\lim_{x \to 3} g(x) = -1$$

$$\lim_{x \to 4^-} g(x) = \lim_{x \to 4}(x - 4) = 4 - 4 = 0$$

$$\lim_{x \to 4^+} g(x) = \lim_{x \to 4} \pi = \pi$$

$$\lim_{x \to 4} g(x) = \text{Does not exist because left-hand and right-hand limits are unequal}$$

$2x - x^2$, $2 - x$, $x - 4$, and π are all continuous on the intervals they're defined on, since they're all polynomial functions. As a result, any points of discontinuity can only occur at the endpoints, $x = 2$ and $x = 3$ and $x = 4$, of the intervals. The function is continuous at $x = 2$ because

$$\lim_{x \to 2} g(x) = g(2) = 0,$$

the function is continuous at $x = 3$ because

$$\lim_{x \to 3} g(x) = g(3) = -1,$$

but the function is discontinuous at $x = 4$ because

$$\lim_{x \to 4} g(x) \neq g(4) = \pi.$$

These results are confirmed in the graph of $g(x)$ versus x.